Lifting Line in Extreme Ground Effect

Prepared for the EAGES 2001 International Ground Effect Symposium
Toulouse, France
June 2001

Kirill V. Rozhdestvensky

D Sc, CEng, FIMarE

Saint Petersburg State Marine Technical University
Russia

Lifting Line in Extreme Ground Effect

Kirill V. Rozhdestvensky

ABSTRACT

The lecture discusses a simplified mathematical model of an ideal fluid flow past a large aspect-ratio wing(s) near a solid underlying surface based on the Ludwig Prandtl's concept of lifting line. With use of the method of matched asymptotic expansions the results are derived in a straightforward manner without solving traditional integro-differential equation(s) for the case of a single lifting line and of a tandem, comprising two wings moving in the same horizontal plane near the ground. The concept implies that the chord of the wing(s) is much smaller than its span. The distance of the lifting line from the ground and longitudinal separation between the wings in the case of the tandem are assumed of the same order as the span. In the limit when relative ground clearance, based on characteristic span of the wing(s), tends to zero, the flow model can be still further simplified through "quadruplication" of relevant integro-differential equations are reduced to ordinary differential equations of the second order. The latter limiting description of a lifting line or lines in extreme ground effect yields simple analytical formulae for aerodynamic coefficients.

ABOUT THE AUTHOR

- Graduated from Leningrad Shipbuilding Institute - LSI (Now Saint-Petersburg State Marine Technical University - SMTU) in 1969.
- 1969 - 1982 : Ph D student, Assistant Professor, Associate Professor of the Department of Hydromechanics of LSI.
- 1982 - present time : Professor, Chairman of the Department of Applied Mathematics and Mathematical Modeling of LSI-SMTU. 1987-1999 Dean of the Faculty of Shipbuilding and Ocean Engineering of SMTU.
- 1999 - present time : Vice Rector for International Science and Education of SMTU.

Author of many publications in the field of :

- Flapping Wing Propulsors
- Hydrodynamics of Wings and Bodies - Hydrodynamics of Ships in Shallow Water
- Asymptotic Methods in Aero-Hydrodynamics of Wings
- Advanced Marine Vehicles
- Wing-in-Ground-Effect Vehicles
- Aerodynamics of Extreme Ground Effect

Three books :

- The Method of Matched Asymptotic Expansions in Hydrodynamics of Wings (Sudostroenie Publishers, Leningrad, 1979)
- Aero-Hydrodynamics of Ships with Dynamic Principles of Support (Text-Book, Sudostroenie Publishers, Leningrad, 1991, Co-Authored by V.K. Treshkov and N.B. Plissov)
- Aerodynamics of a Lifting System in Extreme Ground Effect (Monograph, Springer, Heidelberg-New York-Tokyo, 2000)

Honorary titles and awards :

- Honored Scientist of the Russian Federation (from 2000)
- Recipient of the Denny Gold Medal for the Best Paper of 1996-1997 (The Institute of Marine Engineers, London, paper title "Ekranoplans-the GEMs of Fast Water Transport")
- In 1986-1991 Member of High-Speed Marine Vehicles Committee of the International Towing Tank Conference

INTRODUCTION

In what follows the *ground effect* is understood as an increase in the lift-to-drag ratio of a wing moving close to an underlying surface (ground). The first man-made flying craft to experience this phenomena were aeroplanes employing wings of high aspect ratio. Wright brothers noticed that their gliders covered longer distances when flying close to the sands. Later, in 1910, some pilots reported a peculiarly "different feel"when their aeroplanes flew closer to the ground. This was then referred to in the literature as a "cushioning effect", sometimes leading to a sudden loss of lift. The public perception of the influence of the ground as something more than just an aerodynamic nuisance came when a 56-ton Dornier DO-X seaplane was reported to increase its payload and range during its transatlantic service in the early 30s. As from the first days of aviation the wings of high-aspect-ratio have been seen as most efficient, the developers of wing-in-ground effect vehicles sometimes opted for the design configurations involving such wings. For example, in late 60s Boeing was developing an anti-submarine wing-in-ground effect vehicle known as "Lowboy" (Fig. 1) which had a wing of aspect ratio $\lambda = 12$, i.e. considerably exceeding that of today's commercial aeroplanes ($\lambda = 8 - 10$), [1]. In this project the span of the wing of the 125-ton craft reached 52 meters. A tandem configuration, incorporating two wings of high-aspect ratio was proposed in the 60s by a Swiss engineer Weiland in the course of his contract work with the US company "West Coast". In particular, Weiland was working on a project of a mammoth commercial vehicle with take-off weight of 1000 tons intended to carry up to 3000 passengers, [1]. This twin-hulled vehicle, called "Large Weilandcraft"(Fig. 2), was to have a length of 213 meters and wingspan of 152 meters.

It is common knowledge that Prandtl treated large-aspect-ratio wing as a lifting line with a corresponding trailing vortex sheet, [3]. Introducing notion of effective angle of attack in the far-field and applying Kutta-Zhukovsky condition at the trailing edge of a wing section in the near-field, Prandtl was able to derive an integro-differential equation with respect to span-wise loading distribution. Since Prandtl's famous work, this approach has been used by many authors both for steady and unsteady flow. Van-Dyke [4] viewed relevant flow (boundary) problem as of singular perturbations. He derived a formal asymptotic solution of the problem as an expansion in a small parameter inversely proportional to λ.

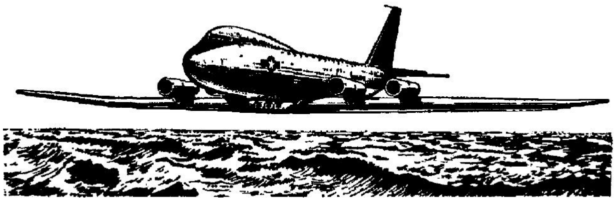

Figure 1 : Ekranoplan Lowboy developed by Boeing in late 60s (project)

A particular feature of Van-Dyke's method was that it permitted to obtain closed expressions for the lift and induced drag coefficients without necessity to solve Prandtl's integro-differential equation. In simple terms, Van-Dyke explicitly employed an observation that induced downwash angle α_i is one order of magnitude smaller than angle of attack α, i.e. $\alpha_i = \mathrm{O}(\alpha/\lambda)$. Later, Van-Dyke's approach was extended to include curved and yawed [5, 6], supercavitating [7], planing [8, 9] and jet-flapped [10] wings of large aspect ratio. Ahmadi and Widnall considered unsteady flow past a wing of large aspect ratio [11]. Tran and Rozhdestvensky [9] applied the same technique to the problem of a wing of large aspect ratio λ, planning upon the surface of weightless fluid of finite depth comparable to the wing's span. To resolve the Green's paradox, consisting in logarithmic behavior of the ordinates of the free surface far upstream and downstream of the wing sections, the authors of [11] matching of the leading order 2D inner solution with the 3D outer solution of the lifting line type as advocated earlier by Shen and Ogilvie for the case of fluid of infinite depth [8]. The influence of the ground upon aerodynamics of a lifting line was investigated by several authors. Panchenkov [12] solved Prandtl's integro-differential equation, extended to the ground-effect case by mirror-image technique, with use of a regular perturbation expansion in terms of a small parameter τ_λ defined as

$$\tau_\lambda = \sqrt{4h_\lambda^2 + 1} - 2h_\lambda \tag{1}$$

where h is ground clearance related to characteristic chord of the wing. Tan and Plotkin [13] analysed a lifting line ground-effect flow, using Glauert method for numerical solution of relevant integro-differential equation. In their lifting line equation authors of both [12] and [13] make use of the slope of the lift coefficient from the corresponding two-dimensional flow problem for a thin-foil in ground effect. Dragos [14] considered more general case of a compressible flow past a thick wing of high-aspect ratio in the spirit of a lifting line theory.

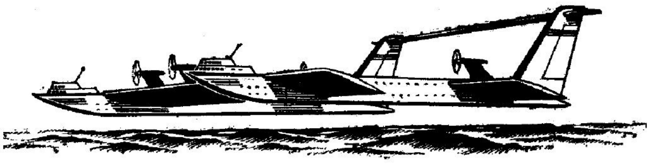

Figure 2 : Transatlantic ekranoplan "Large Weilandcraft" developed by Weiland in the 60s

For a given lift coefficient, the ground-effect phenomena can be characterized by a drag reduction (inverse-efficiency) factor $1/\mu = \lambda/\lambda_{eff}$, where λ_{eff} is effective aspect ratio. Prandtl gave a numerical approximation of this factor for the values he had obtained by numerical integration. Besides, a well-known work of Wieselsberger [15], one can cite cite several other publications, containing estimates of the drag reduction factor, associated with ground effect. That of Suh and Ostowari [16] utilises a simple horse-shoe vortex and image system, see Laitone's correction [17] of [16]. Some useful theoretical and experimental data can be found in Ando [18]. Rozhdestvensky obtained a closed form expression for the inverse efficiency factor based on an asymptotic solution of the problem past a lifting line and a tandem of lifting lines in moderate ($h_\lambda = \mathrm{O}(1)$) and extreme ($h \to 0$) ground effect, [19, 21]. Laitone [20] extended Prandtl's biplane theory to treat wing-tail combinations. In this lecture we will mostly follow the approach applied in Rozhdestvensky [19] and [21]

LIFTING LINE(S) IN MODERATE GROUND EFFECT

The Cartesian coordinate system adopted herein is attached to a reference wing and has z-axis directed to the starboard side, x-axis and y-axis directed downstream and upwards respectively. From now on, unless specified otherwise, all quantities and functions are rendered non-dimensional with use of speed of horizontal motion U_o and semi-span of the reference wing. The ground clearances are measured from the midchord(s) of the wing(s).

Before proceeding to solution of the problem, note that the application of Prandtl's lifting line concept in ground effect aerodynamics implies that characteristic chord of the wing is much smaller than both the span and the ground clearance. Assume first that the ground clearance is of the same order of magnitude as the span (or aspect ratio). Relating all lengths to the span we can express order relationships discussed above as

$$\frac{1}{l} << h_l = \mathrm{O}(1), \tag{2}$$

where l is the ratio of the wing's span to the root chord, h_l is the ground clearance as fraction of the span.

Single Lifting Line in Moderate Ground Effect

For the sake of simplicity we restrict our analysis to consideration of a flat lifting surface. Represent spanwise distribution of local chords by means of equation

$$x = \pm\frac{1}{\lambda}C(z), \tag{3}$$

with the platform function $C(z) = \mathrm{O}(1)$ normalized in such a way that

$$\int_0^1 C(z)\,\mathrm{d}z = 1. \tag{4}$$

With use of the "mirror" image technique the perturbed velocity potential φ_g^o, generated by the lifting line in presence of the ground can be written as

$$\varphi_g^o(x, y, z, h_l) = \varphi(x, y, z) - \varphi(x, y + 4h_l, z), \tag{5}$$

where

$$\varphi(x, y, z) = \frac{1}{4\pi}\int_{-1}^{1}\frac{\Gamma(\zeta)y}{y^2 + (z-\zeta)^2}\Big[1 + \frac{x}{\sqrt{x^2 + y^2 + (z-\zeta)^2}}\Big]\,\mathrm{d}\zeta, \tag{6}$$

Alternatively, the equation (6) can be re-written in the form

$$\varphi(x, y, z) = -\frac{1}{4\pi}\int_{-1}^{1}\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}\Big[\arctan\frac{y}{z-\zeta} + \arctan\frac{y\sqrt{x^2 + y^2 + (z-\zeta)^2}}{x(z-\zeta)}\Big]\,\mathrm{d}\zeta, \tag{7}$$

which may be convenient for evaluation of the behavior of the of the velocity potential in the immediate vicinity of the lifting line without having to tackle divergent integrals. Near the lifting line $x = \mathrm{O}(1/\lambda), y = \mathrm{O}(1/\lambda), z = \mathrm{O}(1)$ the velocity potential can be shown to possess the following asymtotics

$$\begin{gathered}\varphi_g^{oi}(x, y, z) = -\frac{1}{2\pi}\Gamma(z)\arctan\frac{y}{x} - \\ \frac{y}{4\pi}\Big[\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{z-\zeta} + \int_{-1}^{1}\Gamma(\zeta)\frac{(z-\zeta)^2 - 16h_l^2}{[(z-\zeta)^2 - 16h_l^2]^2}\,\mathrm{d}\zeta\Big] - \\ \frac{h_l x}{\pi}\int_{-1}^{1}\frac{\mathrm{d}\zeta}{[(z-\zeta)^2 + 16h_l^2]^{3/2}} - \frac{h_l}{\pi}\int_{-1}^{1}\Gamma(\zeta)\frac{\mathrm{d}\zeta}{(z-\zeta)^2 + 16h_l^2}\end{gathered} \tag{8}$$

Note that the group of terms in (8) proportional to y represents contribution of the velocity potential of the fluid motion, corresponding to downwash induced by the vortex sheet of the lifting line in presence of the ground. This downwash $\alpha_i(z)$ can thus be written as

$$\alpha_i(z) = -\frac{1}{4\pi}\Big\{\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{z-\zeta} + \int_{-1}^{1}\Gamma(\zeta)\frac{(z-\zeta)^2 - 16h_l^2}{[(z-\zeta)^2 - 16h_l^2]^2}\,\mathrm{d}\zeta\Big\} \tag{9}$$

For further use it is convenient to introduce an alternative form of the second term in brackets by performing integration by parts which gives

$$\alpha_i(z) = -\frac{1}{4\pi}\Big\{\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}\Big[\frac{1}{z-\zeta} - \frac{z-\zeta}{(z-\zeta)^2 + 16h_l^2}\Big]\mathrm{d}\zeta\Big\} \tag{10}$$

In the *near field*, i.e. at distances from the wing of the order of the chord one arrives at a 2D formulation for a given wing cross-section z =constant. The outer asymptotic representation of the inner flow potential $\varphi^i \to \varphi^{io}$ behaves as a point vortex in the uniform flow corrected for the downwash of the trailing vortex sheet. The point vortex part of φ^{io} can be expressed as

$$\varphi^{io} \sim -\frac{1}{\pi\lambda}C_{y_{2D}}(z)C(z)\arctan\frac{y}{x} + \ldots \tag{11}$$

where $C_{y_{2D}}$ is the sectional lift coefficient based on the local chord length. Matching of the outer and inner perturbed velocity potentials yields the following leading order spanwise distribution of the circulation of the lifting line

$$\Gamma(z) = \frac{1}{\lambda}C_{y_{2D}}C(z). \tag{12}$$

For a flat wing at angle of attack α the above equation can be reduced to $\Gamma(z) = 2\pi\alpha C(z)/\lambda$ so that the equation (9), describing the induced downwash, can be re-written as

$$\alpha_i(z) = -\frac{\alpha}{2\lambda}\Big\{\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}C(\zeta)}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta}{z-\zeta} + \int_{-1}^{1}C(\zeta)\frac{(z-\zeta)^2 - 16h_l^2}{[(z-\zeta)^2 - 16h_l^2]^2}\,\mathrm{d}\zeta\Big\} \tag{13}$$

Sectional lift coefficient accounting for the downwash can be written as

$$C_{y_s}(z) = C_{y_{2D}}^{\alpha}(z)C(z)[\alpha + \alpha_i(z)] = C_{y_{2D}}C(z)[1 - \frac{1}{\lambda}\bar{\alpha}_i(z)], \tag{14}$$

where $\bar{\alpha}_i(z) = \lambda\alpha_i(z)/\alpha$ does not depend either on the aspect ratio nor on the angle of attack. Integration of the sectional lift coefficient as given by (14) yields the following expression for the lift coefficient of a flat wing of large aspect ratio in a steady motion near the ground plane

$$C_y = 2\pi\alpha\Big[1 - \frac{1}{\lambda}F(h_l)\Big] \sim \frac{2\pi\alpha}{1 + F(h_l)/\lambda}, \tag{15}$$

where function $F(h_l)$ is described by the formula

$$F(h_l) = \frac{1}{4}\Big\{\int_{-1}^{1}C(z)\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}C}{\mathrm{d}\zeta}\frac{\mathrm{d}\zeta\,\mathrm{d}z}{z-\zeta} + \int_{-1}^{1}C(z)\int_{-1}^{1}C(\zeta)\frac{(z-\zeta)^2 - 16h_l^2}{[(z-\zeta)^2 - 16h_l^2]^2}\,\mathrm{d}\zeta\mathrm{d}z\Big\} \tag{16}$$

The sectional induced drag coefficient $C_{x_{i_s}}(z)$ with account of the suction force can be determined with use of the formula

$$C_{x_{i_s}}(z) = 2\pi\frac{\alpha^2}{\lambda}C(z)\Big[1 - \frac{\bar{\alpha}_i(z)}{\lambda}\Big]\bar{\alpha}_i(z). \tag{17}$$

Integration of (17) in spanwise direction results in the expression for the induced drag coefficient of a large-aspect-ratio wing near the ground plane

$$C_{x_i} = \frac{1}{2}\int_{-1}^{1}C_{x_{i_s}}(z)\,\mathrm{d}z = \pi\frac{\alpha^2}{\lambda}\Big[f_1(h_l) - \frac{f_2(h_l)}{\lambda}\Big] \sim \frac{\pi\alpha^2 f_1(h_l)/\lambda}{1 + f_2(h_l)/f_1(h_l)\lambda}. \tag{18}$$

In the above formula the functions $f_{1,2}(h_l)$ are given by the formulae

$$f_1(h_l) = \int_{-1}^{1}C(z)\bar{\alpha}_i(z)\mathrm{d}z, \qquad f_2(h_l) = \int_{-1}^{1}C(z)\bar{\alpha}_i(z)^2\mathrm{d}z \tag{19}$$

Note that both in (15) and (18) we have used an equivalence rule $1 - \eta \sim (1 + \eta)^{-1}$, where η is a small parameter.

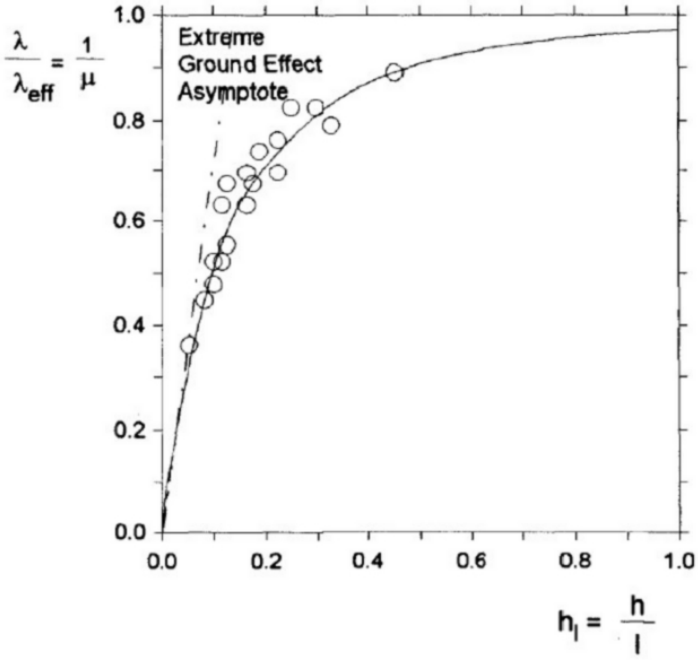

Figure 3 : Inverse efficiency factor $1/\mu = \lambda/\lambda_{eff}$ versus relative ground clearance based on span $h_l = h/l$. Comparison of the present approach and data of Ando [18]

Recalling Prandtl's formula relating lift and induced drag coefficients, we can write

$$C_{x_i} = \frac{C_y^2}{\pi\lambda\mu}, \qquad \mu = \frac{\pi[1 + f_2(h_l)/f_1(h_l)\lambda]}{f_1(h_l)[1 + F(h_l)/\lambda]^2} \tag{20}$$

For a wing of elliptic planform $C(z) = 4\sqrt{1 - z^2}/\pi$ and

$$F(h_l) = 2 + \frac{4}{\pi^2}\int_{-1}^{1}\sqrt{1 - z^2}\int_{-1}^{1}\sqrt{1 - \zeta^2}\frac{(z - \zeta)^2 - 16h_l^2}{[(z - \zeta)^2 - 16h_l^2]^2}\,\mathrm{d}\zeta\mathrm{d}z \tag{21}$$

Figure 3 presents inverse efficiency factor $1/\mu = \lambda/\lambda_{eff}$ (where λ_e is effective aspect ratio) versus relative ground clearance $h_l = h/l$. The plot compares results of the present theory with numerous experimental data referred to by Ando [18] (circles). On the same graph is shown a limiting strait line (dot-and-dash) from a theory of a lifting line in extreme ground effect discussed later.

Tandem of Large Aspect Ratio in Moderate Ground Effect

In this section we shall treat a slightly more complicated lifting system, namely a tandem composed of two wings of large aspect ratio moving in ground proximity. In order to avoid tedious algebra and demonstrate the algorithm of the solution in a simple way our formulation will be restricted to the case when both wings of the tandem have identical ground clearance, planform and aspect ratio. As earlier, all quantities and functions are rendered nondimensional with use of semi-span of the wing and the velocity of the oncoming stream. Perturbed velocity potential of two lifting lines near the ground plane can be constructed as

$$\varphi_{gt}(x, y, z) = \varphi(x, y, z) - \varphi(x, y + 4h_l, z) + \varphi(x - l_t, y, z) - \varphi(x - l_t, y + 4h_l, z) \tag{22}$$

where the function $\varphi(x, y, z)$ is represented by equation (6). Analysis of asymptotic behavior of potential near the front lifting line ($x = \mathrm{O}(y) \to 0$) yields the following expression for the downwash induced on the front wing

$$\alpha_{i_1} = -\frac{1}{4\pi}\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_1(\zeta)}{\mathrm{d}\zeta}\Big[\frac{1}{z-\zeta} - \frac{z-\zeta}{(z-\zeta)^2 + 16h_l^2}\Big]\,\mathrm{d}\zeta +$$

$$\frac{1}{4\pi}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_2(\zeta)}{\mathrm{d}\zeta}\Big[\frac{z-\zeta}{(z-\zeta)^2 + 16h_l^2} + \frac{z-\zeta}{l_t[l_t + \sqrt{l_t^2 + (z-\zeta^2)^2}]} -$$

$$\frac{l_t(z-\zeta)[32h_l^2 + l_t^2 + (z-\zeta)^2]}{(l_t^2 + 16h_l^2)[(z-\zeta)^2 + 16h_l^2]\sqrt{l_t^2 + (z-\zeta)^2 + 16h_l^2}}\Big]\,\mathrm{d}\zeta \qquad (23)$$

Similarly, one can obtain the downwash induced on the rear wing can be derived in the form

$$\alpha_{i_2} = -\frac{1}{4\pi}\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_2(\zeta)}{\mathrm{d}\zeta}\Big[\frac{1}{z-\zeta} - \frac{z-\zeta}{(z-\zeta)^2 + 16h_l^2}\Big]\,\mathrm{d}\zeta +$$

$$\frac{1}{4\pi}\text{v.p.}\int_{-1}^{1}\frac{\mathrm{d}\Gamma_1(\zeta)}{\mathrm{d}\zeta}\Big[\frac{z-\zeta}{(z-\zeta)^2 + 16h_l^2} - \Big[1 + \frac{\sqrt{l_t^2 + (z-\zeta)^2}}{l_t}\Big]\frac{1}{z-\zeta} +$$

$$\frac{l_t(z-\zeta)[32h_l^2 + l_t^2 + (z-\zeta)^2]}{(l_t^2 + 16h_l^2)[(z-\zeta)^2 + 16h_l^2]\sqrt{l_t^2 + (z-\zeta)^2 + 16h_l^2}}\Big]\,\mathrm{d}\zeta \qquad (24)$$

The solution of the flow problem for a tandem of lifting lines in moderate ground effect can be constructed similarly to that for a single lifting line in moderate ground effect. In order to avoid cumbersome derivations in this text, we just refer to some numerical results which were obtained in [19], for a tandem consisting of two flat identical wings with identical (elliptic) platform, adjusted angle of attack, aspect ratio (λ) and ground clearance. Calculated results correspond to relative ground distance between the wings of the tandem equal to $l_t = 7$. Figure 4 shows behaviour of the ratio of the lift coefficient of a tandem near the ground plane $h_l \neq \infty$ to that of the same tandem in unbounded fluid ($h_l = \infty$) versus relative ground clearance. Plotted on the same graph is the corresponding ratio for a single wing. As seen from the Figure, the relative increase of the lift due to ground effect appears to be more pronounced for a tandem than for a single wing. Note, in passing, that though (as shown by Betz) elliptic planform is optimal for a flat-wing tandem out of ground effect, it is no longer optimal in presence of the ground plane. In the following section, dedicated to extreme ground effect ($h_l \to 0$), it is shown that the optimal loading for a tandem of wings of large aspect ratio, advancing in the immediate proximity to the ground, than loading is parabolic rather than elliptic.

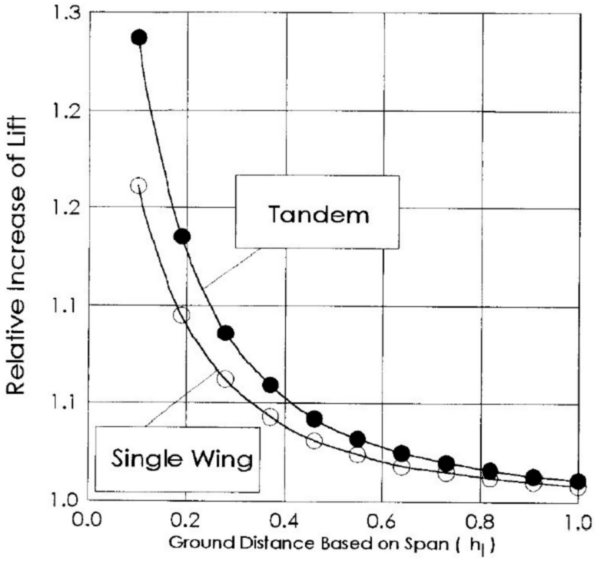

Figure 4 : Lift coefficient of a tandem and a single wing of large aspect ratio (related to the corresponding lift coefficient in unbounded fluid) versus relative ground clearance h_l $(\lambda = 5, l_t = 7)$

LIMITING CASE OF EXTREME GROUND EFFECT

In the preceding paragraphs solutions have been derived for lifting line(s) in moderate ground effect, i.e. in the case when characteristic ground clearance is of the same order of magnitude as reference span of the wing(s). In what follows it will be shown that in the case of the extreme ground effect, i.e. when the ground clearance is much smaller than the reference span of the wing(s) $(h_l << 1)$, it is possible to obtain quite a simple closed solution of the flow problem. In fact, the process $h_l \to 0$ leads to a nontrivial limiting flow which is a result of coming together of lifting line(s) and the corresponding mirror image(s). This resulting flow can be shown to be governed by simple ordinary linear differential equation(s) yielding closed expressions for the lift and moment coefficients both for a single wing and a tandem of large aspect ratio in extreme ground effect.

Single Lifting Line in Extreme Ground Effect

In order to show explicitly how the limiting process is organized in the case of $h_l \to 0$, we first consider a simpler case of a single wing of large aspect ratio in extreme ground effect. Making use of the alternative form of the downwash generated on the wing, one can write Prandtl's integro-differential equation for a flat wing in ground effect in the following form

$$\Gamma(z) = -\frac{2\pi C(z)}{l}\Big\{\alpha(z) - \frac{1}{4\pi}\int_{-1}^{1}\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}\Big[\frac{1}{z-\zeta} - \frac{z-\zeta}{(z-\zeta^2)+16h_l^2}\Big]\,\mathrm{d}\zeta\Big\}, \tag{25}$$

where $\Gamma(z)$, $C(z)$ and $\alpha(z)$ are spanwise distributions of the circulation, local chord and angle of pitch ; l is the relative span (span-to-chord ratio), and $h_l = h/l$ is the clearance-to-span ratio. One can easily see that in the limit $h_l \to 0$ the kernel of the equation (25) tends to zero, which is also a sign of possible instability of numerical solution for very small h_l. However, one can demonstrate that when the limit h_l is applied to the equation (25) properly, the latter can be reduced to a simple ordinary differential equation of the second order.

Split the interval of integration $z \in [-1, 1]$ into three subintervals, thus decomposing the integral into three contributions

$$\int_{-1}^{1} = \int^{z-\eta} + \int_{z-\eta}^{z+\eta} + \int_{z+\eta}^{1} = I_1 + I_2 + I_3, \tag{26}$$

where $\eta \to 0$ is a fictitious small parameter chosen in such a way that for $h_l \to 0$ the ratio η/h_l tends to infinity. It can shown the the contributions of the integrals I_1 and I_3 is of the order of $\mathrm{O}(h_l^2)$, and the main contribution comes from the integral I_2.

Let's consider the latter integral in more detail. Introduce stretching of variables $\zeta - z = h_l Z$, $Z = \mathrm{O}(1)$. Substituting the stretched variables into the integral I_2 and simultaneously expanding $\mathrm{d}\Gamma/\mathrm{d}\zeta$ into Taylor series in the vicinity of the point $\zeta = z$ gives

$$I_2 = -\int_{-\eta/h_l}^{\eta/h_l} \Big[\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}(\zeta) + h_l Z \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\zeta^2} + \ldots\Big] \frac{16}{Z(Z^2+16)}\,\mathrm{d}Z. \tag{27}$$

Applying the limit $h_l \to 0, \eta \to 0, \eta/h_l \to \infty$ we obtain the following important result

$$I_2 = \int_{-\infty}^{\infty} \Big[\frac{\mathrm{d}\Gamma}{\mathrm{d}\zeta}(z) + h_l Z \frac{\mathrm{d}^2\Gamma}{\mathrm{d}\zeta^2}(z) + \ldots\Big] \frac{16}{Z(Z^2+16)}\,\mathrm{d}Z =$$

$$-16 h_l \frac{\mathrm{d}^2\Gamma}{\mathrm{d}z^2}(z) \int_{-\infty}^{\infty} \frac{\mathrm{Z}}{Z^2+16} = -4\pi h_l \frac{\mathrm{d}^2\Gamma}{\mathrm{d}z^2}. \tag{28}$$

Using formula (28) we can reduce the Prandtl's lifting line equation (25) to the following differential equation

$$\Gamma(z) = -\frac{2\pi C(z)}{l}\Big[\alpha(z) - h_l \frac{\mathrm{d}^2\Gamma(z)}{\mathrm{d}z^2}\Big]. \tag{29}$$

Equation (29) should be solved with boundary conditions of zero loading at the tips of the wing, i.e. $\Gamma(\pm 1) = 0$. For further simplicity of the resulting expressions, assume that the wing is not aerodynamically twisted, i.e. $\alpha(z) = \alpha$ is constant.

The equation can be easily integrated for basic spanwise chord distributions. Consider first *the case of a constant chord* $C(z) = 1$ (rectangular wing), for which the resulting expressions for distribution of the circulation along the lifting line and the lift coefficient can be obtained as

$$\Gamma(z) = \frac{2\pi\alpha}{l}\Big[\frac{\cosh(pz)}{\cosh(p)} - 1\Big],$$

$$C_y = 2\pi\alpha\Big[1 - \frac{\tanh(p)}{p}\Big], \qquad p = \sqrt{\frac{l}{2\pi h_l}}. \tag{30}$$

Observing from formula (29) that the downwash in extreme ground effect is proportional to the second derivative of the circulation with respect to the spanwise coordinate z, i.e.

$$\alpha_i(z) = -h_l \frac{\mathrm{d}^2\Gamma}{\mathrm{d}z^2}(z), \tag{31}$$

we can derive the induced drag coefficient for the large-aspect-ratio wing of a rectangular planform in extreme ground effect in the form

$$C_{x_i} = \frac{\pi\alpha^2}{p}\frac{\sinh(2\pi) - 2p}{\cosh(2p) + 1}. \tag{32}$$

As in Prandtl's classical lifting line theory, the induced drag coefficient can be shown to be proportional to the square of the lift coefficient. We can write

$$C_{x_i} = \frac{C_y^2}{\pi\lambda\mu}, \qquad \mu = \frac{\lambda_{eff}}{\lambda} = \frac{4[\cosh(2p)+1][p - \tanh(p)]^2}{\lambda p[\sinh(2p) - 2p]} \tag{33}$$

Examining equation (31), we can conclude that *in the extreme ground effect, the optimal spanwise distribution of loading for a wing of large aspect ratio is parabolic* rather than elliptic, as in the unbounded fluid case. As follows from (28), for a flat untwisted wing, the spanwise chord distribution, providing the parabolic loading distribution, is also parabolic. Substituting $C(z) = \kappa(1 - z^2)$ (where from normalization condition (4) $\kappa = 3/2, \lambda = 3l/2$, and $\Gamma(z) = \Gamma_o(1 - z^2)$) into equation (28), we obtain the following formula for Γ_o

$$\Gamma_o = \frac{2\pi\alpha}{l(1 + 4\pi h_l/l)}. \tag{34}$$

To obtain the lift coefficient based on the chord, we have to apply the following formula

$$C_y = \Gamma_o l \int_{-1}^{1} (1 - z^2)\, \mathrm{d}z = \frac{8\pi\alpha}{3(1 + 4\pi h_l/l)}. \tag{35}$$

The downwash, corresponding to the parabolic spanwise loading is constant along the lifting line. Simple calculation shows that

$$\alpha_i(z) = -h_l \frac{\mathrm{d}^2\Gamma}{\mathrm{d}z^2}(z) = -2h_l\Gamma_o = -\frac{8h_l\pi\alpha}{1 + 4\pi h_l/l}. \tag{36}$$

The induced drag coefficient of the optimal lifting line is calculated as the lift coefficient times the induced downwash, i.e.

$$C_{x_i} = C_y\alpha_i = \frac{8h_l l\Gamma_o}{3} = \frac{3h_l}{2l} C_y^2 \tag{37}$$

or, rewriting (37) in the Prandtl's format

$$C_{x_i} = \frac{C_y^2}{\pi\lambda\mu}, \qquad \mu = \frac{\lambda_{eff}}{\lambda} = \frac{4}{9\pi h_l}. \tag{38}$$

These results show that in the limiting flow problem for a lifting line in extreme ground effect, the effective aspect ratio is inversely proportional to the ground clearance related to span. Figure 5 presents the inverse efficiency factor $1/\mu$ versus the relative ground clearance (based on the span) for a single wing with rectangular and parabolic platforms of the same relative span $l = 5$, operating in extreme ground effect. Figure 5 was obtained with use of the formulae (33) and (38).

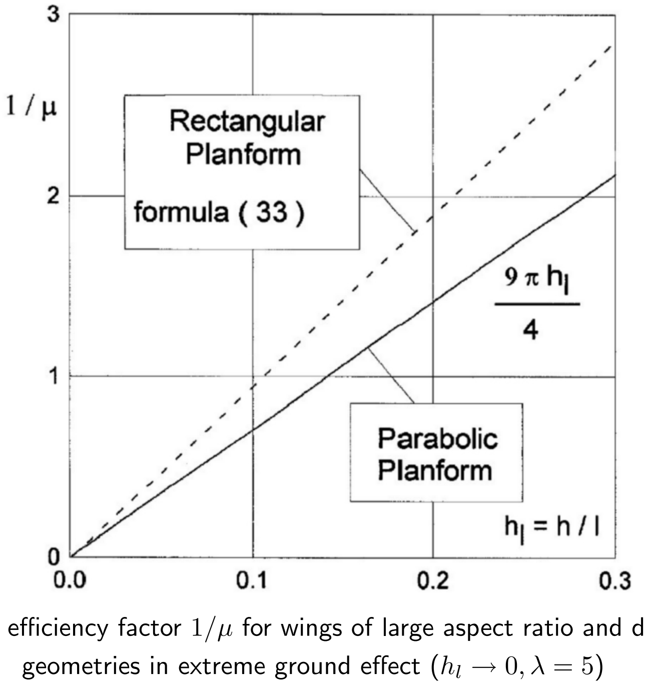

Figure 5 : Inverse efficiency factor $1/\mu$ for wings of large aspect ratio and different planform geometries in extreme ground effect ($h_l \to 0, \lambda = 5$)

A Tandem of Lifting Lines in the Extreme Ground Effect

Similar limiting procedures for $h_l \to 0$ applied to the Prandtl's system of integro-differential equations for a tandem consisting of two identical flat large-aspect-ratio wings, moving at the same ground clearance, yield the following set of two ordinary linear differential equations of the second order, see Rozhdestvensky [19, 21]

$$\Gamma_1(z) = -\frac{2\pi C(z)}{\lambda}\Big[\alpha_1(z) - h_l\frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}z^2}(z)\Big], \tag{39}$$

$$\Gamma_2(z) = -\frac{2\pi C(z)}{\lambda}\Big[\alpha_2(z) - h_l\frac{\mathrm{d}^2\Gamma_2}{\mathrm{d}z^2}(z) - 2h_l\frac{\mathrm{d}^2\Gamma_1}{\mathrm{d}z^2}(z)\Big] \tag{40}$$

In these equations $\Gamma_{1,2}$ stand for the distribution of the loading in the direction of span of the front and rear wings of the tandem, $C(z)$ represent the form of the chord distribution, and $\alpha_{1,2}$ are spanwise distributions of the angle of attack.

It can be seen from observation of the right-hand side of equation (39) that in the extreme ground effect, the downwash induced by the rear wing upon the front wing is negligible. At the same time, the front wing affects the aerodynamics of the rear wing, see equation (40).

Let's pass over to concrete platforms. Suppose that both wings are of *rectangular planform*, i.e. C(z)=1, and flat (untwisted) $\alpha_1(z) = \alpha_1$, $\alpha_2(z) = \alpha_2$. Then, the first equation of the system (39)-(40) can be integrated to give

$$\Gamma_1(z) = \frac{2\pi\alpha_1}{\lambda}\Big[\frac{\cosh(pz)}{\cosh(p)} - 1\Big], \qquad p = \sqrt{\frac{\lambda}{2\pi h_l}} = \sqrt{\frac{l}{2\pi h_l}}. \tag{41}$$

Substituting solution for $\Gamma_1(z)$ in the second equation of the system (39)-(40), we obtain the following nonhomogeneous ordinary differential equation for the function $\Gamma_2(z)$

$$\frac{\mathrm{d}^2\Gamma_2(z)}{\mathrm{d}z^2} - p^2\Gamma_2(z) = \frac{\alpha_2}{h_l} + \frac{2\alpha_1\cosh(pz)}{h_l\cosh(p)}, \tag{42}$$

Integrating (42) and using the requirement that the loading should vanish at the wing tips $\Gamma_2(\pm 1) = 0$, we obtain the following solution for Γ_2

$$\Gamma_2 = \Big[A - \frac{\alpha_1\cosh(2pz)}{2p^2\cosh(p)}\Big]\cosh(pz) +$$
$$\frac{\alpha_1}{ph_l\cosh(p)}\Big[\frac{\sinh(2pz)}{2p} + 1\Big]\tanh(p) + \frac{\alpha_2}{h_l p^2\cosh(p)}. \tag{43}$$

Using expressions (41) and (43) for the loading along the front and rear wings, we can readily obtain both the lift and the induced drag coefficients for the above case of a rectangular platform of the lifting elements of the tandem.

However, in what follows, the accent will be on a *parabolic loading distribution* for which each of the wings and the tandem as a whole have minimal induced drag for a given lift. Writing the circulations and the platform equations of both lifting lines as $\Gamma_{1,2} = \Gamma_{o1,2}(z^2 - 1)$, $C_{1,2} = \kappa(1 - z^2), \kappa = 3/2$ and substituting these expressions into equations (39) and (40), we come to the following simple system of algebraic equations with respect to the amplitudes of the loading distributions

$$\Gamma_{10} = \frac{2\pi}{\lambda}(\alpha_1 - 2h_l\Gamma_{10}), \qquad \Gamma_{20} = \frac{2\pi}{\lambda}(\alpha_2 - 2h_l\Gamma_{20} - 4h_l\Gamma_{10}) \tag{44}$$

wherefrom

$$\Gamma_{10} = \frac{2\pi\alpha_1}{l(1 + 4\pi h_l/l)}, \qquad \Gamma_{20} = \frac{2\pi(\alpha_2 - 4h_l\Gamma_{10})}{l(1 + 4\pi h_l/l)}. \tag{45}$$

The lift coefficients of each wing and the overall lift coefficient of the tandem C_{y_t} are obtained in the form

$$C_{y_1} = \frac{4l\Gamma_{10}}{3} = \frac{8\pi\alpha_1}{l(1+4\pi h_l/l)}, \tag{46}$$

$$C_{y_2} = \frac{4l\Gamma_{20}}{3} = \frac{8\pi(\alpha_2 - 4h_l\Gamma_{10})}{l(1+4\pi h_l/l)}. \tag{47}$$

$$C_{y_t} = C_{y_1} + C_{y_2}. \tag{48}$$

In the latter coefficient, the reference area used was half that of the tandem. For optimal wing loading, the downwash on both lifting lines is uniform along the span

$$\alpha_{i_1} = 2h_l\Gamma_{10}, \qquad \alpha_{i_2} = 2h_l\Gamma_{20} + 4h_l\Gamma_{10}. \tag{49}$$

Consequently, the induced drag coefficients for the front and rear wings can be found in the form

$$C_{x_{i_1}} = C_{y_1}\alpha_1 = \frac{3h_l}{2l}{Cy_1}^2, \quad C_{x_{i_2}} = C_{y_2}\alpha_2 = \frac{3h_l}{2l}{Cy_2}^2 + \frac{3h_l}{l}C_{y_1}C_{y_2}. \tag{50}$$

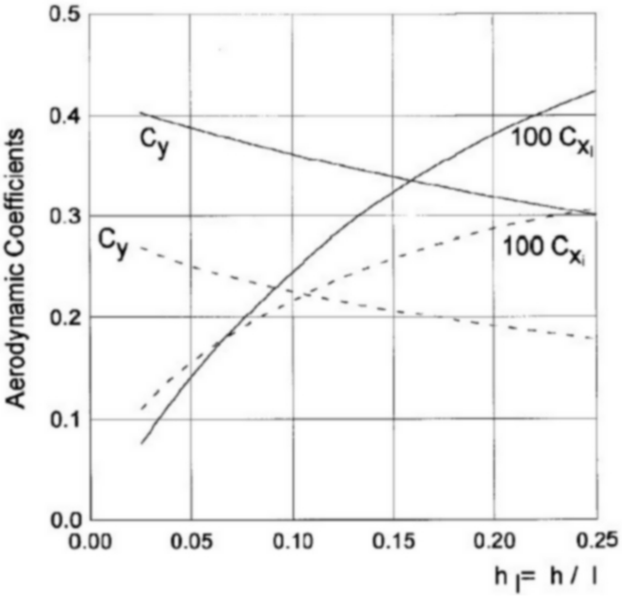

Figure 6 : Behaviour of the lift and induced drag coefficients versus relative ground clearance based on span (SDGE model) in extreme ground effect ($l = 8, \theta = 0.05$), solid lines correspond to a parabolic planform, dashed lines correspond to a rectangular planform

The induced drag of the tandem as a whole will be

$$C_{x_i} = C_{x_{i_1}} + C_{x_{i_2}} = \frac{3h_l}{2l}(C_{y_1}^2 + 2C_{y_1}C_{y_2} + C_{y_2}^2) = \frac{3h_l}{2l}(C_{y_1} + C_{y_2})^2. \tag{51}$$

It is worthwhile to recall that the optimum tandem in unbounded fluid ($h_l = \infty$) has the following relationship between the lift and the induced drag coefficients

$$C_{x_{i_1}} + C_{x_{i_2}} = \frac{1}{\pi\lambda}(C_{y_1} + C_{y_2})^2. \tag{52}$$

Figure 6 illustrates behaviour of the lift and drag coefficients of a single large-aspect-ratio wing versus relative ground clearance based on span in extreme ground effect ($l = 8, \alpha = 0.05$) for parabolic and rectangular platforms. We see that for a fixed angle of attack and decreasing ground clearance the lift coefficient increases whereas the induced drag coefficient decreases. Such a dependence of induced drag on ground clearance can be expected for situations when the chord is small compared to the distance from the ground, and the latter is smaller than the span. In [21] it is associated with "span-dominated ground effect".

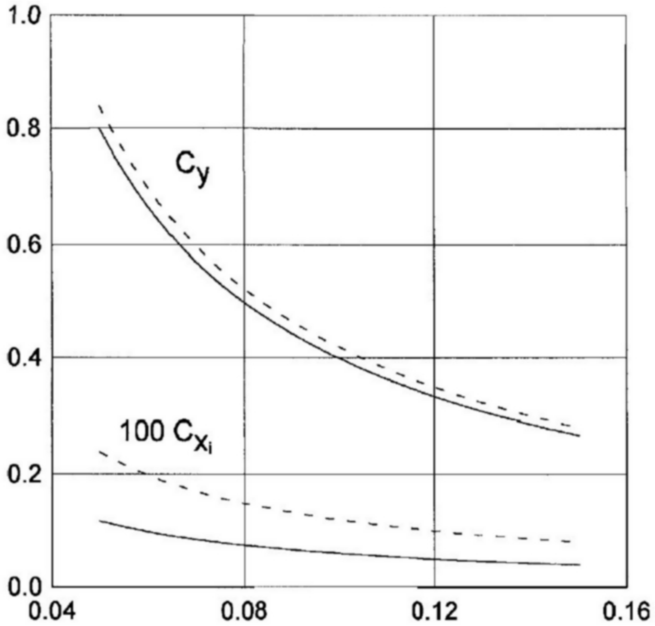

Figure 7 : Behaviour of the lift and induced drag coefficients versus relative ground clearance based on chord (CDGE model) in extreme ground effect ($l = 8, \theta = 0.05$), solid lines correspond to a parabolic planform, dashed lines correspond to a rectangular planform

Simultaneously, it is shown in [21] that when the distance from the ground is smaller than the chord of the wing, the decrease of the clearance brings about growth of both the lift and induced drag coefficients, see Fig. 7 from [21]. The latter effect can be called "chord dominated ground effect"[1]. Note that Standingford and Tuck [22] came to the same conclusions in their accurate numerical analysis of aerodynamics of thin lifting surfaces for small ratios of ground clearance to the chord.

As recalled by Synitsin [23], the father of Russian ekranoplans Rostislav Alekseev used to speak about a "ground effect in drag" and "ground effect in lift".

REFERENCES

1. Belavin, N.I. "Ekranoplans", *Sudostroenie*, Leningrad, 1968 and 1977.
2. Rozhdestvensky, K.V. "Ekranoplans - The GEMs of Fast Water Transport", *Transactions of the Institute of Marine Engineers*, London, Vol. 109, Part 1, pp. 47-74, 1997.
3. Prandtl, L. "Wing Theory", *Bulletin of the Götingen Scientific Society*, 1919.
4. Van Dyke, M.D. "Lifting Line Theory as a Singular Perturbation Problem", *Appl. Math. Mech.*, 28, pp. 90-101, 1964.

---

[1] Note that this discussion is based on calculations for flat and vanishingly thin wings

5. Dorodnitsin, A.A. “Generalization of Lifting Line Theory to the Cases of a Wing with Curved Axis and Axis Non-Normal to the Flow”, *Prikladnaya Matematika i Mekhanika*, Vol. VIII, No. 1, 1944.
6. Cheng, H.K. “Lifting Line Theory of Oblique Wings”, *AIAA Journal*, Vol. 16, pp. 1211-1213.
7. Furuya, H. “Three-Dimensional Theory on Supercavitating Hydrofoils near a Free Surface”, *Journal of Fluid Mechanics*, Vol. 71, Part 2, pp. 39-359, 1975.
8. Shen, Y.T. and Ogilvie, T.F. “Nonlinear Hydrodynamic Theory for Finite-Span Planing Surfaces”, *Journal of Ship Research*, Vol. 14, pp. 3-20, 1972.
9. Tran, N.N. and Rojdestvenskii, K.V. “Hydroplanage d’une Plaque Plane de Grande Envergure sur la Surface d’un Domaine de Fluide de Profondeur Finie”, *Journal de Mecanique*, Vol. 14, No. 5, 1975.
10. Tokuda, N. “An Asymptotic Theory of the Jet Flap in Three-Dimensions”, *Journal of Fluid Mechanics*, Vol. 46, Part 4, pp. 705-726, 1971.
11. Ahmadi, A.R. and Widnall, S.E. “Unsready Lifting Line as a Singular Perturbation Problem”, *Journal of Fluid Mechanics*, Vol. 153, pp. 59-81, 1975.
12. Panchenkov, A.N. “Hydrodynamics of an Underwater Wing”, *Naukova Dumka*, Kiev, 552 p., 1965.
13. Tan, C.N., Plotkin, A. “Lifting Line Solution for a Symmetrical Thin Wing in Ground Effect”, *AIAA Journal*, No. 24, pp. 1193-1194, 1986.
14. Dragos, L. “Subsonic Flow Past Thick Wing in Ground Effect - Lifting Line Theory”,*Acta Mechanica*, No. 82, pp. 49-60, 1990.
15. Wieselsberger, C. “Wing Resistance near the Ground”, *NACA TM77*, 1922.
16. Suh, Y.B. and Ostowari, C. “Drag Reduction Due to Ground Effect”, *Journal of Aircraft*, Vol. 25, No. 11, pp. 1071-1072, 1988.
17. Laitone, E.V. “Comment on ‘Drag Reduction Factor due to Ground Effect”’, *Journal of Aircraft*, Vol. 27, No. 1, p. 96, 1989.
18. Ando, S. “Note on Prediction of Aerodynamic Lift-to Drag Ratio of WIG at Cruise”, *FAST’93*, Vol. 2, pp. 1657-1670, 1993.
19. Rozhdestvensky, K.V. “Aerodynamics of a Tandem of High Aspect Ratio in Ground Effect”, *Proceedings of the International Workshop ‘WISE up to Ekranoplan GEMs’, The University of South Wales, Sydney*, 15-16 June 1998, pp. 166-184.
20. Laitone, E.V. “Extension of Prandtl’s Biplane Theory to Wing-Tail Combinations”, *Canadian Aeronautics and Space Journal*, Vol. 25, No. 3, pp. 278-285, 1979.
21. Rozhdestvensky, K.V. “Aerodynamics of Lifting System in Extreme Ground Effect”, *Springer*, Heidelberg-New York-Tokyo, 352 p., 2000.
22. Standingford, D.W.F., Tuck, E.O. “Lifting Surfaces in Ground Effect”, *Proc. of the International Workshop “Ekranoplans and Very Fast Craft”*, the University of New South Wales, Sydney, Australia, 5-6 December 1996, pp. 230-243.
23. Synitsin, D.N. Private communication to the author, October, 1998.

DISCUSSION

Bernard Masure (BM), Université d'Orléans

I would like to comment your last remarks on lift and drag[2]. I think you should compare the L/D.

Kirill V. Rozhdestvensky (KVR), Saint Petersburg State Marine Technical University

That's correct. We have in this case compared lift and drag on a common basis. In one case, we have chord related ground clearance and on the other one span related ground clearance. If we compare strong RAM effect with span dominated ground effect, the L/D goes up when distances diminish.

Chairman Allan Bonnet (AB), SUPAERO

One could add that maybe it is not very relevant to compare the L/D as the viscous drag is not taken into account.

KVR

Speaking about viscous effects, of course, as usually in the lifting line, engineers all do the same : add the friction or viscous drag. You can use some kind of classical turbulent boundary layer flat plate formula. It will give the maximum L/D you can have for a given configuration of the wing. This certainly depends proportionally to the square of the friction drag coefficient. But this can be different depending on what is the effective aspect ratio. So this model gives the possibility of simple estimates when one understands fairly well what are the geometric configurations and what craft configurations one considers.

Pr. Kirill Vsevolodovitch Rozhdestvensky

Chairman Allan Bonnet (AB), SUPAERO

I'd like to make a comment on one of the formulae. The one that gives the downwash[3]. This is really interesting and maybe students will be surprised. You see that this quantity depends on h_l^2 and the limiting process only depends on h_l. When the limiting process is done correctly, we find that something that we thought depended on h_l to a given power finally depends on simply h_l !

KVR

It is true. One can also organize the same process for a lifting surface and its mirror image. And

---

[2] Figure 6 and 7. The Editor

[3] Equation (25) to (36). The Editor

if one considers a limiting process where $h \to 0$, either based on span or chord because they are of the same order of magnitude, then one can show, when doing accurately this limiting process, that this is again a quadruplication. So what one gets from a lifting surface when $h \to 0$, is in linear a Poisson equation in 2D. It is again, instead of an integral singular equation, the degenerated problem of a differential equation. This is again this channel flow problem. We have 3D in free air and we have 2D closer to the ground because everything is determined by the channel flow where everything moves in longitudinal plans and is restricted to move in the vertical direction.

Henri Charles Ozarovsky (HCO), Euroavia Toulouse

What is the optimum way of finishing the wings ? You know, on regular airplanes you have winglets. . .

KVR

Here the best solution is fitting endplates because for a given span the endplates make the wing effective aspect ratio bigger. But I would say that as the span increases, you will need less and less the endplates. On the contrary, when you reduce the aspect ratio, you reduce your efficiency unless you put endplates or do something else like providing a local curvature to the wing. As far as the winglets are concerned, I would say that all the devices that fight with parasite vortices that are useful in aviation are useful here. Maybe to less extent because of the cancelling phenomenon related to ground effect. But I think one should try any solution to reduce induced drag that is the main component of the drag here.